REMARKS ON HARMONIC MAPS, SOLITONS, AND DILATON GRAVITY

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Another connection of harmonic maps to gravity is presented. Using 1-soliton and anti-soliton solutions of the sine-Gordon equation, we construct a pair of harmonic maps that we express in terms of a particular dilaton field in Jackiw-Teitelboim gravity. This field satisfies a linearized sine-Gordon equation. We use it also to construct an explicit transformation that relates the corresponding solitonic metric to a two dimensional black hole metric.

1 Introduction

The theory of harmonic maps provides a pleasant, unifying setting in which various field equations can be viewed and discussed. The equations of motion of a Bosonic string, for example, coincide with the requirement that the map of its world sheet to 26-dimensional space should be harmonic. The solution of the $O(3)$ σ-model is provided by a harmonic map from the unit 2-sphere S^2 to itself. Certain Einstein equations are given by harmonic maps. A broad overview of the role of harmonic maps in Yang-Mills theory, general relativity, and quantum field theory is presented in the inspiring papers of C. Misner[4] and N. Sánchez[5], for example.

We consider a pair of harmonic maps from the plane R^2 to S^2 that we relate to sine-Gordon solitons and to two dimensional Jackiw-Teitelboim dilaton gravity. An intriguing observation of J. Gegenberg and G. Kunstatter connects such solitions with two dimensional black holes[2,3]. We construct an explicit transformation Ψ that takes the solitonic metric to a black hole metric, and we express the harmonic maps in terms of the dilaton field.

2 Harmonic maps from R^2 to S^2

A smooth map $\Phi : (M,g) \to (N,h)$ of Riemannian (or pseudo Riemannian) manifolds is called harmonic if it satisfies the following (local) conditions; one can also formulate harmonicity by a global condition[1]. Let $(U,\phi = (x_1,\ldots,x_m))$ and $(V,\psi = (y_1,\ldots,y_n))$ be coordinate systems on M,N with $U \subset \Phi^{-1}(V)$, let $\Phi^j = y_j \circ \Phi \circ \phi^{-1}(1 \le j \le n)$ denote the j^{th} component of Φ relative to these systems, let Γ_{ij}^k denote the Christoffel symbols

of (N, h), and let $\partial_i = \frac{\partial}{\partial x_i} (1 \leq i \leq m)$. If

$$B = \frac{1}{\sqrt{|\det(g \circ \phi^{-1})|}} \sum_{i,j=1}^{m} \partial_i \sqrt{|\det(g \circ \phi^{-1})|} (g^{ij} \circ \phi^{-1}) \partial_j \qquad (1)$$

is the Laplacian of (M, g) on $\phi(U)$, then we require that

$$(\tilde{B}_s \Phi)(p) \equiv \sum_{i,j=1}^{m} (g^{ij} \circ \phi^{-1}) \sum_{k,r=1}^{n} \partial_i \Phi^k \partial_j \Phi^r \Big|_{\phi(p)} \Gamma^s_{kr}(\Phi(p)) + (B\Phi^s) \Big|_{\phi(p)} = 0 \qquad (2)$$

for $p \in U$, $1 \leq s \leq n$. We construct harmonic maps $\Phi = \Phi^{\pm} : R^2 \to S^2$ as follows. $\Phi = (\cos\beta \sin\alpha, \ \sin\beta \sin\alpha, \ \cos\alpha)$ where $\alpha, \ \beta : R^2 \to R$ are given by $\alpha(x,t) = u(x,t)/2, \ \ \beta(x,t) = m(vx+t)/a$ for parameters $m, v > 0, a = a(v) \equiv \sqrt{1+v^2}$ where for $\rho(x,t) \equiv m(x - vt)/a, u(x,t) = u^{\pm}(x,t) = 4\tan^{-1} e^{\pm\rho(x,t)}$ are one-soliton solutions of the Euclidean sine-Gordon equation (SGE)

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = m^2 \sin u. \qquad (3)$$

The harmonicity condition on Φ in fact reduces to condition (3) - which contrasts the point of view in [3] where (3) is obtained by variation of the dilaton field τ in the Jackiw-Teitelboim (J-T) action

$$I(\tau, u) = \frac{1}{2G} \int dx \int dt \ \tau \left[\Delta u - m^2 \sin u\right]. \qquad (4)$$

m is revealed as a mass parameter and v as a soliton velocity parameter. As pointed out in [3] the linearised SGE $\Delta\tau = (m^2 \cos u)\tau$ is satisfied by the field

$$\tau(x,t) = a(v) \text{sech} \rho(x,t). \qquad (5)$$

3 Satement of the main result

The solitonic metric

$$ds^2 = \cos^2 \alpha(x,t) dx^2 - \sin^2 \alpha(x,t) dt^2 \qquad (6)$$

has scalar curvature $R = \frac{2\Delta u}{\sin u}$, which is therefore constant by (3): $R = 2m^2$; or the Gaussian curvature $K = -\frac{R}{2} = -m^2$. Recall that $a = a(v) = \sqrt{1+v^2}$.

Theorem

Let $\Psi = (\psi_1, \psi_2) : R^2 \to R^2$ be the transformation defined as follows:

$$\begin{aligned} \psi_1(T,r) &= vT + \tfrac{1}{m} \coth^{-1} \left[\sqrt{a^2 - m^2 r^2}\right] \ , \\ \psi_2(T,r) &= \tfrac{\psi_1(T,r)}{v} - \tfrac{a}{mv} \log \left[\tfrac{a + \sqrt{a^2 - m^2 r^2}}{mr}\right] \end{aligned} \qquad (7)$$

on the domain

$$C^\dagger = \left\{ (T,r) \in R^2 \middle| \, 0 < r < \frac{a}{m}, \sqrt{a^2 - m^2 r^2} > 1 \right\}, \tag{8}$$

and let $\Theta = (\theta_1, \theta_2)) : R^2 \to R^2$ be defined by

$$\begin{aligned} \theta_1(x,t) &= -\tfrac{1}{mv}\coth^{-1}[a\ \tanh\rho(x,t)] + \tfrac{x}{v}, \\ \theta_2(x,t) &= \tfrac{a}{m}\mathrm{sech}\rho(x,t) = \tfrac{\tau(x,t)}{m} \end{aligned} \tag{9}$$

on the domain

$$D^\dagger = \left\{ (x,t) \in R^2 \middle| \, a\ |\tanh\rho(x,t)| > 1 \right\}. \tag{10}$$

Then $\Psi : C^\dagger \to D^\dagger$ and $\Theta : D^\dagger \to C^\dagger$ are bijections and inverses of each other: $\Theta \circ \Psi = 1$ on $C^\dagger$, $\Psi \circ \Theta = 1$ on $D^\dagger$. Also Ψ transforms the solitonic metric (6) to the black hole metric

$$ds^2 = (M - m^2 r^2) dT^2 - (M - m^2 r^2)^{-1} dr^2 \tag{11}$$

with mass $M = v^2$. Thus, conversely, Θ takes (11) back to (6). Also the harmonic maps $\Phi^\pm$ constructed in the preceding section are expressed in terms of the dilaton field (5) as follows:

$$\Phi^\pm = \frac{1}{a}(\tau\cos\beta,\ \tau\sin\beta, \pm a\ \tanh\rho). \tag{12}$$

The interesting connection of sine-Gordon solitons to black hole solitons in J-T gravity is the remarkable observation of the paper [3], although the transformation Ψ that we have presented here does not explicitely appear there. We have considered another connection of harmonic maps to gravity.

References

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